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Fifth Monthly Progress Report

on

Thermal Strain Analysis

of

Advanced Manned-Spacecraft Heat Shields

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SUMMARY

The summary contained herein for work accomplished during the last monthly period represents effort in the following four phases of the program:

Phase A'	Formulation of Boundary Conditions
Phase A"	Investigation of Engineering Models
Phase F	Reduction to Axially Symmetric Case and Formulation of Sample Problem
Phase G	Preparation of Reports and Computer Program for Delivery

Phase A' - Formulation of Boundary Conditions

In the previous Monthly Progress Report, all of the boundary and interface conditions required in the heat shield analysis were described and formulated, with the exception of the thin-shell conditions which were incomplete at that time. These conditions were, however, described in general terms in Equations (10) and (11) of Reference 1, using the functions f_1 , f_2 , ... f_8 , which were then undefined. With the completion of the thin-shell analysis, which is included in the Appendix of this report, the boundary condition formulation is essentially complete. It is noted that Equations (17) in the Appendix, in toroidal curvilinear coordinates, correspond with the second and third equations of Equation (10) in Reference 1, which were written using Cartesian coordinate notation. Also, Equation (26) in the Appendix corresponds with the first of Equation (10) in Reference 1, where P_z

in Equation (26) corresponds with the stress difference $\sigma_z^{(1)} - \sigma_z^{(2)}$ in Equation (10), Reference 1. The stress-displacement conditions indicated by Equation (1') in Reference 1, in Cartesian notation, may be written explicitly in toroidal coordinates according to

$$(\sigma_z)_i = (\lambda_i + 2\mu_i)\left(\frac{\partial u}{\partial r}\right)_i + \frac{\lambda_i (a + 2r\sin\varphi)}{r(a + r\sin\varphi)} u_i + \frac{\lambda_i}{r}\left(\frac{\partial v}{\partial \varphi}\right)_i + \frac{\lambda_i \cos\varphi}{a + r\sin\varphi} v_i + \frac{\lambda_i}{a + r\sin\varphi}\left(\frac{\partial w}{\partial \theta}\right)_i - (3\lambda_i + 2\mu_i)\int_{T_0}^{T} \alpha_i(T)\, dT$$

$$(\tau_{r\varphi})_i = \frac{\mu_i}{r}\left(\frac{\partial u}{\partial \varphi}\right)_i + \mu_i\left(\frac{\partial v}{\partial r}\right)_i - \frac{\mu_i}{r} v_i$$

$$(\tau_{r\theta})_i = \frac{\mu_i}{a + r\sin\varphi}\left(\frac{\partial u}{\partial \theta}\right)_i + \mu_i\left(\frac{\partial w}{\partial r}\right)_i - \frac{\mu_i \sin\varphi}{a + r\sin\varphi} w_i$$

where the subscript i identifies the two media adjoining the thin shell; i.e., $i = 1, 2$. The six equations above plus the three equations defined in Equations (17) and (26) of the Appendix completely define the thin-shell interface conditions required for the heat shield analysis. At the capping surface (Surface 3 in Figure 1 of Reference 1), the neutral surface displacements u, v and w of the thin shell must satisfy the same conditions imposed on the displacements of the "thick-shell" regions at this surface, namely, $u = v = w = 0$ for the fixed-edge condition and $\tau_{\varphi\varphi} = \tau_{r\varphi} = \tau_{\varphi\theta} = 0$ for the free edge condition. In the case of the thin shell, the latter condition on stresses reduces to the condition $N_\varphi = M_\varphi = M_{\varphi\theta} = 0$ where the sectional forces and moments are defined in Equation (29) of the Appendix. This gives three equations in the three neutral surface displacements at each node lying in the capping surface.

In reviewing the formulation of boundary and interface conditions previously presented in Appendix 1 of Reference 1, it was noted that the geometric juncture at the sphere-torus interface was treated as a physical interface, and the boundary conditions were specified accordingly. This treatment, although not incorrect, is more cumbersome than necessary for this type of interface. A better approach makes use of an averaging procedure defined as follows:

The central differences with respect to φ, which span the boundary between the two geometric regions are differenced as if they were wholly within one region, (e.g., the toroidal region, denoted Region 1). Since this involves function values in the spherical region (denoted Region 2) which are not defined in terms of the coordinate grid of Region 1, the r = constant lines of Region 1 which are involved are extended into Region 2. These lines will be approximately congruent with the corresponding R = constant lines of Region 2, for incremental distances from the geometric juncture. The "extended" node is chosen on the extended grid line to be the same distance (along the grid line) from the geometric juncture as the actual Region 2 node. This choice of the increment in φ between the "extended" node and the juncture node, independent of that between the juncture node and that lying just inside Region 1, is possible because of the use of "irregular" difference approximations adopted in the φ direction. The actual Region 2 node is thereby "close" to the "extended" node. The displacement function values defined on the former are therefore carried over to the latter.

Similarly, a second equation is obtained by differencing as if the point and its neighborhood were wholly in Region 2. A linear combination of the two equations obtained in Regions 1 and 2, respectively, is taken to be the best approximation to the difference analog at the juncture.

Phase A" - Investigation of Engineering Models

The basic approach to the problem of treating very thin layers in a composite heat shield were set forth in the Third Monthly Progress Report, Reference 5. For simplicity, the method was presented for a flat plate, using Cartesian coordinates. In the Fourth Monthly, Reference 1, this analysis was generalized to the case of spherical curvilinear coordinates but was not completed. Because of the similarity of toroidal and spherical coordinates and the fact that spherical coordinates are a limiting case of toroidal coordinates, the equations were rederived in the Appendix of this report using toroidal coordinates and including temperature dependence of the elastic constants. The analysis is complete except for presenting the final results in tabular form in terms of coefficients of the equations in terms of displacements, as was done previously for the equilibrium and stress equations.

Phase F - Reduction to Axially Symmetric Case and Formulation of Sample Problem

A portion of this work was completed in conjunction with verifying the correctness of the three-dimensional equations but was not reported in previous monthly progress reports. For example, [illegible] to establish the validity of the coefficients of the equilibrium equations, reported in Reference 2, it was verified in the case of spherical coordinates that these coefficients reduce to the axially symmetric forms derived by A. J. A. Morgan in his study "Thermal Stresses in Missile Nose Cones" (Reference 3). Since spherical coordinates are a limiting case of toroidal coordinates, the reduction to the axially symmetric case is also a check on the validity of the equilibrium equations in toroidal coordinates. Similarly, the stress displacement relations, which were derived and tabulated in the Second Monthly Progress Report (Reference 4), were also shown to reduce to those derived by Morgan for the axially symmetric case. The axis of symmetry was shown to require special treatment in the non-axially symmetric case owing to singularities which occur in the equilibrium equations as the coordinate φ approaches zero. In the axially symmetric case, the singularities can be handled by the the use of L'Hôspital's rule. The coefficients of the equilibrium equations for this case were presented in Tables 4 and 5 of Reference 4. It was verified in the case of spherical coordinates that these coefficients agree with those derived by Morgan for this special case.

In summary, the conditions for axial symmetry require that

$$w(R, \varphi, \theta) = \frac{\partial f(R, \varphi, \theta)}{\partial \theta} = 0 \quad , \tag{1}$$

where w is the azimuthal component of the displacement vector in the θ-direction (see Figure 1, Reference 2) and f is any function of the coordinates. On the axis of symmetry ($\varphi = 0$) it can be shown that

$$v = \frac{\partial u}{\partial \varphi} = \frac{\partial^2 v}{\partial \varphi^2} = 0 \; . \tag{2}$$

If the conditions of Equation (1) are applied to the equilibrium-displacement equations (Equations (16) of Reference 2), these equations reduce to

$$\begin{aligned} & A_k \frac{\partial^2 u}{\partial \alpha_1^2} + B_k \frac{\partial^2 u}{\partial \alpha_2^2} + D_k \frac{\partial^2 u}{\partial \alpha_1 \partial \alpha_2} + G_k \frac{\partial u}{\partial \alpha_1} + H_k \frac{\partial u}{\partial \alpha_2} \\ & + J_k u + \bar{A}_k \frac{\partial^2 v}{\partial \alpha_1^2} + \bar{B}_k \frac{\partial^2 v}{\partial \alpha_2^2} + \bar{D}_k \frac{\partial^2 v}{\partial \alpha_1 \partial \alpha_2} + \bar{G}_k \frac{\partial v}{\partial \alpha_1} \\ & + \bar{H}_k \frac{\partial v}{\partial \alpha_2} + \bar{J}_k v = \frac{(3\lambda + 2\mu)\,\alpha(T)}{\sqrt{g_{kk}}} \frac{\partial T}{\partial \alpha_k}, \quad k = 1, 2, \end{aligned} \tag{3}$$

where α_1, and α_2 are the curvilinear coordinates R or r and φ respectively. These coefficients in terms of the Lame' constants λ and μ (from Table 1, Reference 2), are given in Table 1, along with the corresponding coefficients from Table IV, Reference 3, expressed in terms of Young's modulus and Poisson's ratio. It is verified from the expressions relating these elastic constants

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \; , \quad \mu = \frac{E}{2(1+\nu)} \; , \tag{4}$$

that the coefficients A_k, B_k, etc., are related to the coefficients $A^{(k)}$, $B^{(k)}$, etc. according to

$$A_k/\mu = A^{(k)} \; , \quad B_k/\mu = B^{(k)} \; , \quad etc. \tag{5}$$

Table 1. Coefficients of Displacement-Equilibrium Equations in Spherical Coordinates with Axial Symmetry

	(From Table 1, Ref. 2) $k=1$	$k=2$		(From Table IV, Ref. 3.) $k=1$	$k=2$
$\frac{A_k}{\mu}$	$\frac{\lambda+2\mu}{\mu}$	0	$A^{(k)}$	$\frac{2(1-\nu)}{1-2\nu}$	0
$\frac{B_k}{\mu}$	$\frac{1}{R^2}$	0	$B^{(k)}$	$\frac{1}{R^2}$	0
$\frac{D_k}{\mu}$	0	$\frac{\lambda+\mu}{\mu R}$	$C^{(k)}$	0	$\frac{1}{(1-2\nu)R}$
$\frac{G_k}{\mu}$	$\frac{2(\lambda+2\mu)}{R\mu}$	0	$D^{(k)}$	$\frac{4(1-\nu)}{(1-2\nu)R}$	0
$\frac{H_k}{\mu}$	$\frac{\cot\varphi}{R^2}$	$\frac{2(\lambda+2\mu)}{\mu R^2}$	$E^{(k)}$	$\frac{\cot\varphi}{R^2}$	$\frac{4(1-\nu)}{(1-2\nu)R^2}$
$\frac{J_k}{\mu}$	$-\frac{2(\lambda+2\mu)}{\mu R^2}$	0	$F^{(k)}$	$-\frac{4(1-\nu)}{(1-2\nu)R^2}$	0
$\frac{\bar{A}_k}{\mu}$	0	1	$\bar{A}^{(k)}$	0	1
$\frac{\bar{B}_k}{\mu}$	0	$\frac{\lambda+2\mu}{\mu R^2}$	$\bar{B}^{(k)}$	0	$\frac{2(1-\nu)}{(1-2\nu)R^2}$
$\frac{\bar{D}_k}{\mu}$	$\frac{\lambda+\mu}{\mu R}$	0	$\bar{C}^{(k)}$	$\frac{1}{(1-2\nu)R}$	0
$\frac{\bar{G}_k}{\mu}$	$\frac{(\lambda+\mu)\cot\varphi}{\mu R}$	$\frac{2}{R}$	$\bar{D}^{(k)}$	$\frac{\cot\varphi}{(1-2\nu)R}$	$\frac{2}{R}$
$\frac{\bar{H}_k}{\mu}$	$-\frac{(\lambda+3\mu)}{\mu R^2}$	$\frac{(\lambda+2\mu)\cot\varphi}{\mu R^2}$	$\bar{E}^{(k)}$	$-\frac{(3-4\nu)}{(1-2\nu)R^2}$	$\frac{2(1-\nu)\cot\varphi}{(1-2\nu)R^2}$
$\frac{\bar{J}_k}{\mu}$	$-\frac{(\lambda+3\mu)\cot\varphi}{\mu R^2}$	$-\frac{(\lambda+2\mu)}{\mu R^2\sin^2\varphi}$	$\bar{F}^{(k)}$	$-\frac{(3-4\nu)\cot\varphi}{(1-2\nu)R^2}$	$-\frac{2(1-\nu)}{(1-2\nu)R^2\sin^2\varphi}$

Similarly, it can be shown that the coefficients of the equilibrium equations in toroidal coordinates, from Table 2, Reference 2, correspond with the coefficients of the equilibrium equations in biconical coordinates from Table IV, Reference 3. Figure 1 shows the relationship between the two coordinate systems. In order to

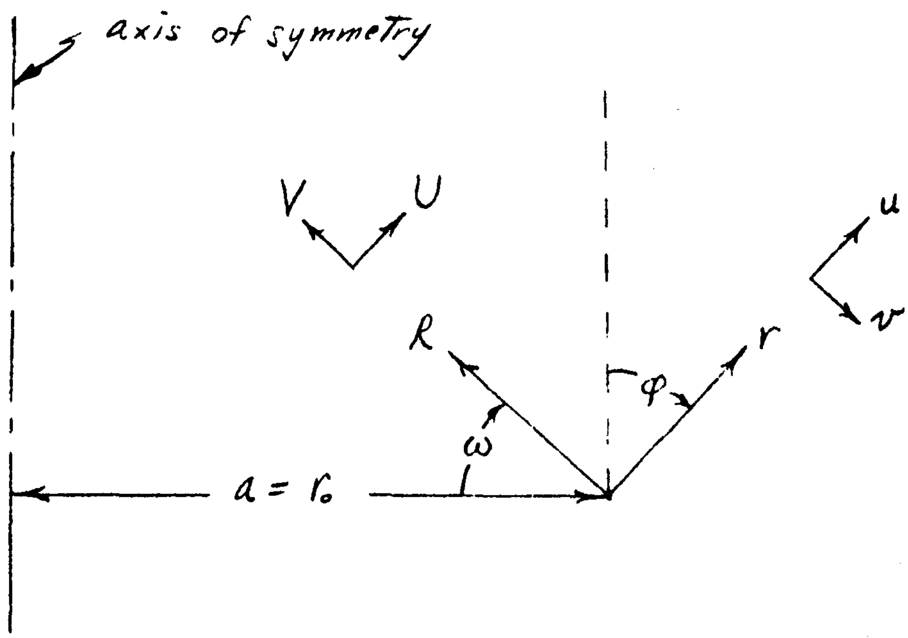

Figure 1. - Toroidal and Biconical Coordinates

compare the two systems, it should be noted that the displacements U and V in biconical coordinates, in the direction of increasing ω and R, respectively, correspond with the displacements v and u in toroidal coordinates in the direction of increasing φ and r, respectively. Also, the reference points for the angles φ and ω differ by 90° such that $\varphi = \omega - \pi/2$. The index k, which identifies two equilibrium equations in the two coordinate directions, takes on different values in the two coordinate systems; i.e., $k=1$ in biconical coordinates corresponds with $k=2$ in toroidal coordinates, and vice-versa.

Taking into account the differences noted above, the corresponding coefficients of the equilibrium equations in the two coordinate systems are given in Table 2. As in the case of spherical coordinates, the coefficients in toroidal coordinates are related to those in biconical coordinates through the constant factor μ, according to Equation (5).

It can be readily verified that the two sets of coefficients are in agreement by making the following substitutions in accordance with the above discussion:

$$\left.\begin{aligned}
\sin\varphi &= \sin(\omega - \pi/2) = -\cos\omega \\
\cos\varphi &= \cos(\omega - \pi/2) = \sin\omega \\
a &\rightarrow r_0 \\
r &\rightarrow R \\
\frac{\lambda+2\mu}{\mu} &= \frac{2(1-\nu)}{1-2\nu} \\
\frac{\lambda+3\mu}{\mu} &= \frac{3-4\nu}{1-2\nu} \\
\frac{\lambda+\mu}{\mu} &= \frac{1}{1-2\nu}
\end{aligned}\right\} \qquad (6)$$

To illustrate, consider, for example, the coefficient $\bar{J}_2/\mu$ in toroidal coordinates and the corresponding coefficient $F^{(1)}$ in biconical coordinates. Rewriting $\bar{J}_2/\mu$, there is obtained

Table 2. Coefficients of Displ... Equilibrium Equations in Toroidal and Biconic... Coordinates with Axial Symmetry

	Toroidal Coordinates (From Ref. 2, Table 2) $k=1$	$k=2$		Biconical Coordinates (From Ref. 3, Table IV) $k=2$	$k=1$
$\frac{\bar{B}_k}{\mu}$	0	$\frac{\lambda+2\mu}{\mu r^2}$	$A^{(k)}$	0	$\frac{2(1-\nu)}{(1-2\nu)R^2}$
$\frac{\bar{A}_k}{\mu}$	0	1	$B^{(k)}$	0	1
$\frac{\bar{D}_k}{\mu}$	$\frac{(\lambda+\mu)}{\mu r}$	0	$C^{(k)}$	$\frac{1}{(1-2\nu)R}$	0
$\frac{\bar{H}_k}{\mu}$	$-\frac{(\lambda+3\mu)}{\mu r^2}$	$\frac{(\lambda+2\mu)\cos\varphi}{\mu r(a+r\sin\varphi)}$	$D^{(k)}$	$-\frac{(3-4\nu)}{(1-2\nu)R^2}$	$\frac{2(1-\nu)\sin\omega}{(1-2\nu)R(r_0-R\cos\omega)}$
$\frac{\bar{G}_k}{\mu}$	$\frac{(\lambda+\mu)\cos\varphi}{\mu(a+r\sin\varphi)}$	$\frac{(a+2r\sin\varphi)}{r(a+r\sin\varphi)}$	$E^{(k)}$	$\frac{\sin\omega}{(1-2\nu)(r_0-R\cos\omega)}$	$\frac{r_0-2R\cos\omega}{R(r_0-R\cos\omega)}$
$\frac{\bar{J}_k}{\mu}$	$-\left[\frac{(\lambda+3\mu)r\sin\varphi\cos\varphi+\mu a\cos\varphi}{\mu r(a+r\sin\varphi)^2}\right]$	$-\left[\frac{(\lambda+2\mu)r^2+\mu a^2+(\lambda+3\mu)ar\sin\varphi}{\mu r^2(a+r\sin\varphi)^2}\right]$	$F^{(k)}$	$-\left[\frac{(1-2\nu)r_0-(3-4\nu)R\cos\omega}{1-2\nu}\right]\frac{\sin\omega}{R(r_0-R\cos\omega)^2}$	$-\frac{1}{R^2}\left[1+\frac{R^2\sin^2\omega}{(r_0-R\cos\omega)^2}\right]+\frac{1}{(1-2\nu)R}\left[\frac{r_0\cos\omega-R}{(r_0-R\cos\omega)^2}\right]$
$\frac{B_k}{\mu}$	$\frac{1}{r^2}$	0	$\bar{A}^{(k)}$	$\frac{1}{R^2}$	0
$\frac{A_k}{\mu}$	$\frac{\lambda+2\mu}{\mu}$	0	$\bar{B}^{(k)}$	$\frac{2(1-\nu)}{1-2\nu}$	0
$\frac{D_k}{\mu}$	0	$\frac{\lambda+\mu}{\mu r}$	$\bar{C}^{(k)}$	0	$\frac{1}{(1-2\nu)R}$
$\frac{H_k}{\mu}$	$\frac{\cos\varphi}{r(a+r\sin\varphi)}$	$\frac{2(\lambda+2\mu)r\sin\varphi+(\lambda+3\mu)a}{\mu r^2(a+r\sin\varphi)}$	$\bar{D}^{(k)}$	$\frac{\sin\omega}{R(r_0-R\cos\omega)}$	$\frac{2}{R^2}+\frac{1}{(1-2\nu)R^2}\left[\frac{r_0-2R\cos\omega}{r_0-R\cos\omega}\right]$
$\frac{G_k}{\mu}$	$\frac{(\lambda+2\mu)(a+2r\sin\varphi)}{\mu r(a+r\sin\varphi)}$	0	$\bar{E}^{(k)}$	$\frac{2(1-\nu)}{(1-2\nu)R}\left[\frac{r_0-2R\cos\omega}{r_0-R\cos\omega}\right]$	0
$\frac{J_k}{\mu}$	$-\frac{(\lambda+2\mu)}{\mu}\left[\frac{1}{r^2}+\frac{\sin^2\varphi}{(a+r\sin\varphi)^2}\right]$	$\frac{(\lambda+2\mu)a\cos\varphi}{\mu r(a+r\sin\varphi)^2}$	$\bar{F}^{(k)}$	$-\frac{2(1-\nu)}{(1-2\nu)R^2}\left[1+\frac{R^2\cos^2\omega}{(r_0-R\cos\omega)^2}\right]$	$\frac{2(1-\nu)r_0\sin\omega}{(1-2\nu)R(r_0-R\cos\omega)^2}$

Table 2. Coefficients of Displacement-Equilibrium Equations in Toroidal and Biconical Coordinates with Axial Symmetry

$$\bar{J}_2/\mu = -\frac{1}{(a+r\sin\varphi)^2}\left[\frac{\lambda+2\mu}{\mu} + \frac{a^2}{r^2} - \frac{(\lambda+3\mu)\,a\sin\varphi}{\mu r}\right]$$

$$= -\frac{1}{(r_0-R\cos\omega)^2}\left[\frac{2(1-\nu)}{1-2\nu} + \frac{r_0^2}{R^2} - \frac{3-4\nu}{1-2\nu}\cdot\frac{r_0\cos\omega}{R}\right].$$

The coefficient $F^{(2)}$ may be rewritten as follows:

$$F^{(2)} = -\frac{1}{R^2}\left[1+\frac{R^2\sin^2\omega}{(r_0-R\cos\omega)^2}\right] + \frac{1}{(1-2\nu)R}\left[\frac{r_0\cos\omega - R}{(r_0-R\cos\omega)^2}\right]$$

$$= -\frac{1}{(r_0-R\cos\omega)^2}\left[\frac{(r_0-R\cos\omega)^2}{R^2} + \sin^2\omega - \frac{r_0\cos\omega}{(1-2\nu)R} + \frac{1}{1-2\nu}\right]$$

$$= -\frac{1}{(r_0-R\cos\omega)^2}\left[\frac{r_0^2-2r_0R\cos\omega+R^2\cos^2\omega}{R^2} + \sin^2\omega - \frac{r_0\cos\omega}{(1-2\nu)R} + \frac{1}{1-2\nu}\right]$$

$$= -\frac{1}{(r_0-R\cos\omega)^2}\left[\frac{r_0^2}{R^2} - \frac{(3-4\nu)}{1-2\nu}\frac{r_0\cos\omega}{R} + \frac{2(1-\nu)}{1-2\nu}\right]$$

It is seen that the two coefficients are identical. It can be verified in a similar manner that all of the corresponding coefficients of Table 2 are equal.

It was shown in Reference 4 that certain of the coefficients in spherical coordinates become singular on the axis of symmetry ($\varphi = 0$). These coefficients were evaluated by the use of L'Hôspital's rule and are presented in Table 4, Reference 4. It can be seen that these results are in agreement with those calculated by Morgan (see Table IV, Reference 3) for the corresponding coefficients indicated in Table 1.

In addition to the equations of equilibrium, the stress-displacement relations must also reduce to those calculated by Morgan for the axially symmetric case. These coefficients, for the general case, are given in Tables 3 and 4, Reference 4. In the case of axial symmetry, the conditions of Equation (1) must be satisfied, which cause certain of the coefficients to vanish. The stress-displacement relations for the general case, from Equation (18), Reference 4, are given by

$$
\begin{aligned}
\tau_\ell + \Delta_\ell (3\lambda+2\mu)\int_{T_0}^{T} \alpha(T)\,dT = {} & \alpha_\ell \mu_r + \beta_\ell \mu_\varphi + \gamma_\ell u_\theta + \delta_\ell u \\
& + \bar{\alpha}_\ell v_r + \bar{\beta}_\ell v_\varphi + \bar{\gamma}_\ell v_\theta + \bar{\delta}_\ell v \\
& + \bar{\bar{\alpha}}_\ell w_r + \bar{\bar{\beta}}_\ell w_\varphi + \bar{\bar{\gamma}}_\ell w_\theta + \bar{\bar{\delta}}_\ell w \,,
\end{aligned}
\tag{7}
$$

where the subscripts r, φ and θ denote differentiation,

$$
\begin{aligned}
\Delta_\ell &= 1 \quad \text{if} \quad \ell = 1,2,3 \\
&= 0 \quad \text{if} \quad \ell = 4,5,6
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_1 &= \tau_{rr} \text{ or } \tau_{RR} \\
\tau_2 &= \tau_{\varphi\varphi} \\
\tau_3 &= \tau_{\theta\theta} \\
\tau_4 &= \tau_{r\varphi} \text{ or } \tau_{R\varphi} \\
\tau_5 &= \tau_{\varphi\theta} \\
\tau_6 &= \tau_{r\theta} \text{ or } \tau_{R\theta}\,.
\end{aligned}
$$

This expression corresponds with Equation (14), Reference 3, for the axially symmetric case, which is

$$
\begin{aligned}
G^{(\ell)}\sigma_\ell + H^{(\ell)}\int_{T_0}^{T} \alpha(\bar{T})\,d\bar{T} = {} & \alpha^{(\ell)} \frac{\partial U}{\partial \theta^1} + \beta^{(\ell)} \frac{\partial U}{\partial \theta^2} + \gamma^{(\ell)} U \\
& + \bar{\alpha}^{(\ell)} \frac{\partial V}{\partial \theta^1} + \bar{\beta}^{(\ell)} \frac{\partial V}{\partial \theta^2} + \bar{\gamma}^{(\ell)} V, \quad \ell = 1,2,3,4,
\end{aligned}
\tag{8}
$$

where

$$\sigma_1 = \sigma_{11} = \sigma_{\rho\rho} \quad or \quad \sigma_{\omega\omega}$$

$$\sigma_2 = \sigma_{22} = \sigma_{\varphi\varphi} \quad or \quad \sigma_{RR}$$

$$\sigma_3 = \sigma_{33} = \sigma_{\phi\phi}$$

$$\sigma_4 = \sigma_{12} = \sigma_{\rho\varphi} \quad or \quad \sigma_{\omega R} \quad .$$

It is seen from a comparison of Equations (7) and (8) that the coefficients α_ℓ, β_ℓ, ... are related to the coefficients $\alpha^{(\ell)}$, $\beta^{(\ell)}$, ... according to

$$G^{(\ell)}\alpha_\ell = \alpha^{(\ell)} , \quad etc., \tag{9}$$

where the constant $G^{(\ell)}$ is given by

$$G^{(\ell)} = \frac{(1+\nu)(1-2\nu)}{E} = \frac{1}{2(\lambda+\mu)} \quad , \quad \ell = 1, 2, 3$$

$$= \frac{2(1+\nu)}{E} = \frac{1}{\mu} \quad , \quad \ell = 4 .$$

It is also noted, in the case of toroidal and biconical coordinates, since u and v correspond with V and U, respectively, and φ corresponds with ω (see Figure 1), that the barred quantities in toroidal coordinates correspond with the unbarred quantities in biconical coordinates and the α's and β's and the indices 1 and 2 are interchanged. The equivalent coefficients from Table 3, Reference 4 and Table V, Reference 3 are compared in Tables 3 and 4. Using the relations between the elastic constants

$$\left.\begin{aligned} \frac{\lambda}{2(\lambda+\mu)} &= \nu \\ \frac{\lambda+2\mu}{2(\lambda+\mu)} &= 1-\nu \end{aligned}\right\} \qquad (10)$$

and the expressions of Equations 6, it is readily verified that the corresponding coefficients in Tables 3 and 4 are identical.

Table 3. Coefficients of Stress-Displacement Equations in Spherical Coordinates with Axial Symmetry

	(From Table 3, Ref. 4)					(From Table V, Ref. 3)			
	$l=1$	$l=2$	$l=3$	$l=4$		$l=1$	$l=2$	$l=3$	$l=4$
$G^{(l)}\alpha_l$	$\frac{\lambda+2\mu}{2(\lambda+\mu)}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{\lambda}{2(\lambda+\mu)}$	0	$\alpha^{(l)}$	$1-\nu$	ν	ν	0
$G^{(l)}\beta_l$	0	0	0	$\frac{1}{R}$	$\beta^{(l)}$	0	0	0	$\frac{1}{\rho}$
$G^{(l)}\delta_l$	$\frac{\lambda}{R(\lambda+\mu)}$	$\frac{1}{R}$	$\frac{1}{R}$	0	$\gamma^{(l)}$	$\frac{2\nu}{\rho}$	$\frac{1}{\rho}$	$\frac{1}{\rho}$	0
$G^{(l)}\bar{\alpha}_l$	0	0	0	1	$\bar{\alpha}^{(l)}$	0	0	0	1
$G^{(l)}\bar{\beta}_l$	$\frac{\lambda}{2R(\lambda+\mu)}$	$\frac{\lambda+2\mu}{2R(\lambda+\mu)}$	$\frac{\lambda}{2R(\lambda+\mu)}$	0	$\bar{\beta}^{(l)}$	$\frac{\nu}{\rho}$	$\frac{1-\nu}{\rho}$	$\frac{\nu}{\rho}$	0
$G^{(l)}\bar{\delta}_l$	$\frac{\lambda \cot\varphi}{2R(\lambda+\mu)}$	$\frac{\lambda \cot\varphi}{2R(\lambda+\mu)}$	$\frac{(\lambda+2\mu)\cot\varphi}{2R(\lambda+\mu)}$	$-\frac{1}{R}$	$\bar{\gamma}^{(l)}$	$\frac{\nu \cot\varphi}{\rho}$	$\frac{\nu \cot\varphi}{\rho}$	$\frac{(1-\nu)\cot\varphi}{\rho}$	$-\frac{1}{\rho}$

Table 4. Coefficients of Stress-Displacement Equations in Toroidal and Biconical Coordinates with Axial Symmetry

	Toroidal Coordinates (From Table 3, Ref. 4)					Biconical Coordinates (From Table V, Ref. 3)			
	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$		$\ell = 2$	$\ell = 1$	$\ell = 3$	$\ell = 4$
$G^{(\ell)}_{\alpha_\ell}$	$\frac{\lambda + 2\mu}{2(\lambda+\mu)}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{\lambda}{2(\lambda+\mu)}$	0	$\bar{\beta}^{(\ell)}$	$1-\nu$	ν	ν	0
$G^{(\ell)}_{\beta_\ell}$	0	0	0	$\frac{1}{r}$	$\bar{\alpha}^{(\ell)}$	0	0	0	$\frac{1}{R}$
$G^{(\ell)}_{\delta_\ell}$	$\frac{\lambda(a+2r\sin\varphi)}{2r(\lambda+\mu)(a+r\sin\varphi)}$	$\frac{\lambda+2\mu}{2r(\lambda+\mu)} + \frac{\lambda\sin\varphi}{2(\lambda+\mu)(a+r\sin\varphi)}$	$\frac{\lambda}{2r(\lambda+\mu)} + \frac{(\lambda+2\mu)\sin\varphi}{2(\lambda+\mu)(a+r\sin\varphi)}$	0	$\bar{\gamma}^{(\ell)}$	$\frac{\nu}{R}\left[\frac{r_0 - 2R\cos\omega}{r_0 - R\cos\omega}\right]$	$\frac{1}{R}\left[\frac{(1-\nu)r_0 - R\cos\omega}{r_0 - R\cos\omega}\right]$	$\frac{1}{R}\left[\frac{\nu r_0 - R\cos\omega}{r_0 - R\cos\omega}\right]$	0
$G^{(\ell)}_{\bar{\alpha}_\ell}$	0	0	0	1	$\beta^{(\ell)}$	0	0	0	1
$G^{(\ell)}_{\bar{\beta}_\ell}$	$\frac{\lambda}{2r(\lambda+\mu)}$	$\frac{\lambda+2\mu}{2r(\lambda+\mu)}$	$\frac{\lambda}{2r(\lambda+\mu)}$	0	$\alpha^{(\ell)}$	$\frac{\nu}{R}$	$\frac{1-\nu}{R}$	$\frac{\nu}{R}$	0
$G^{(\ell)}_{\bar{\delta}_\ell}$	$\frac{\lambda\cos\varphi}{2(\lambda+\mu)(a+r\sin\varphi)}$	$\frac{\lambda\cos\varphi}{2(\lambda+\mu)(a+r\sin\varphi)}$	$\frac{(\lambda+2\mu)\cos\varphi}{2(\lambda+\mu)(a+r\sin\varphi)}$	$-\frac{1}{r}$	$\gamma^{(\ell)}$	$\frac{\nu\sin\omega}{r_0 - R\cos\omega}$	$\frac{\nu\sin\omega}{r_0 - R\cos\omega}$	$\frac{(1-\nu)\sin\omega}{r_0 - R\cos\omega}$	$-\frac{1}{R}$

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Appendix

Thin-Shell Equations in Toroidal Coordinates

with

Temperature - Dependent Elastic Constants

Thin-Shell Equations in Toroidal Coordinates

The toroidal coordinate system is described by the equation for the line element

$$ds^2 = \alpha_1^2 dr^2 + \alpha_2^2 d\varphi^2 + \alpha_3^2 d\theta^2 \tag{1}$$
$$= dr^2 + r^2 d\varphi^2 + (a + r\sin\varphi)^2 d\theta^2 ,$$

where

$$\alpha_1 = 1 , \quad \alpha_2 = r , \quad \alpha_3 = a + r\sin\varphi . \tag{2}$$

The middle surface strains, ϵ_{φ_0}, ϵ_{θ_0}, $\gamma_{\varphi\theta_0}$, and changes of curvature, χ_φ, χ_θ, $\chi_{\varphi\theta}$, are written in terms of the middle surface displacements, u, v, w (in the r, φ and θ directions, respectively) and principal radii of curvature, R_2 and R_3, according to (Ref. 1)

$$\left.\begin{aligned}
\epsilon_{\varphi_0} &= \frac{1}{\alpha_2}\frac{\partial v}{\partial\varphi} + \frac{w}{\alpha_2\alpha_3}\frac{\partial\alpha_2}{\partial\theta} + \frac{u}{R_2} \\
\epsilon_{\theta_0} &= \frac{1}{\alpha_3}\frac{\partial w}{\partial\theta} + \frac{v}{\alpha_2\alpha_3}\frac{\partial\alpha_3}{\partial\varphi} + \frac{u}{R_3} \\
\gamma_{\varphi\theta_0} &= \frac{\alpha_3}{\alpha_2}\frac{\partial}{\partial\varphi}\left(\frac{w}{\alpha_3}\right) + \frac{\alpha_2}{\alpha_3}\frac{\partial}{\partial\theta}\left(\frac{v}{\alpha_2}\right) \\
\chi_\varphi &= \frac{1}{\alpha_2}\frac{\partial}{\partial\varphi}\left(\frac{v}{R_2} - \frac{1}{\alpha_2}\frac{\partial u}{\partial\varphi}\right) + \frac{1}{\alpha_2\alpha_3}\left(\frac{w}{R_3} - \frac{1}{\alpha_3}\frac{\partial u}{\partial\theta}\right)\frac{\partial\alpha_2}{\partial\theta} \\
\chi_\theta &= \frac{1}{\alpha_3}\frac{\partial}{\partial\theta}\left(\frac{w}{R_3} - \frac{1}{\alpha_3}\frac{\partial u}{\partial\theta}\right) + \frac{1}{\alpha_2\alpha_3}\left(\frac{v}{R_2} - \frac{1}{\alpha_2}\frac{\partial u}{\partial\varphi}\right)\frac{\partial\alpha_3}{\partial\varphi} \\
\chi_{\varphi\theta} &= \frac{1}{2}\left[\frac{\alpha_3}{\alpha_2}\frac{\partial}{\partial\varphi}\left(\frac{2w}{\alpha_3 R_3} - \frac{1}{\alpha_3^2}\frac{\partial u}{\partial\theta}\right) + \frac{\alpha_2}{\alpha_3}\frac{\partial}{\partial\theta}\left(\frac{v}{\alpha_2 R_2} - \frac{1}{\alpha_2^2}\frac{\partial u}{\partial\varphi}\right)\right]
\end{aligned}\right\} \tag{3}$$

Substituting α_1, α_2 and α_3 from Eq. (2) in Eq. (3) and differentiating as indicated, there is obtained for the middle surface strains and changes of curvature*

* The principal radii of curvature are

$$R_2 = r , \qquad R_3 = \frac{a + r\sin\varphi}{\sin\varphi}$$

$$\left.\begin{aligned}
\epsilon_{\varphi_0} &= \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r} \\
\epsilon_{\theta_0} &= \frac{1}{a+r\sin\varphi}\frac{\partial w}{\partial \theta} + \frac{\cos\varphi}{a+r\sin\varphi}\, v + \frac{\sin\varphi}{a+r\sin\varphi}\, u \\
\gamma_{\varphi\theta_0} &= \frac{a+r\sin\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{w}{a+r\sin\varphi}\right) + \frac{r}{a+r\sin\varphi}\frac{\partial}{\partial\theta}\left(\frac{v}{r}\right) \\
&= \frac{1}{r}\frac{\partial w}{\partial\varphi} - \frac{\cos\varphi}{a+r\sin\varphi}\, w + \frac{1}{a+r\sin\varphi}\frac{\partial v}{\partial\theta} \\
\chi_\varphi &= \frac{1}{r^2}\frac{\partial v}{\partial\varphi} - \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} \\
\chi_\theta &= \frac{\sin\varphi}{(a+r\sin\varphi)^2}\frac{\partial w}{\partial\theta} - \frac{1}{(a+r\sin\varphi)^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\varphi}{r(a+r\sin\varphi)}\left(v - \frac{\partial u}{\partial\varphi}\right) \\
\chi_{\varphi\theta} &= \frac{1}{2}\Bigg[\frac{2\cos\varphi\cdot w}{r(a+r\sin\varphi)} - \frac{4\sin\varphi\cos\varphi}{(a+r\sin\varphi)^2}\, w + \frac{1}{r(a+r\sin\varphi)}\frac{\partial v}{\partial\theta} \\
&\qquad - \frac{2}{r(a+r\sin\varphi)}\frac{\partial^2 u}{\partial\varphi\,\partial\theta} + \frac{2\cos\varphi}{(a+r\sin\varphi)^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\varphi}{r(a+r\sin\varphi)}\frac{\partial w}{\partial\varphi}\Bigg]
\end{aligned}\right\} \quad (4)$$

According to the Kirchhoff-Love hypothesis for thin shells, the strain components through the shell thickness are given by

$$\left.\begin{aligned}
\epsilon_\varphi &= \epsilon_{\varphi_0} - z\,\chi_\varphi \\
\epsilon_\theta &= \epsilon_{\theta_0} - z\,\chi_\theta \\
\gamma_{\varphi\theta} &= \gamma_{\varphi\theta_0} - 2z\,\chi_{\varphi\theta}
\end{aligned}\right\} \quad (5)$$

Where z is measured along the negative r-direction from the neutral surface which will be defined later. Substituting Eqs. (4) in Eqs. (5), the strain components become

$$\epsilon_\varphi = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r} - z\left(\frac{1}{r^2}\frac{\partial v}{\partial \varphi} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right)$$

$$\epsilon_\theta = \frac{1}{a+r\sin\varphi}\frac{\partial w}{\partial \theta} + \frac{\cos\varphi}{a+r\sin\varphi}v + \frac{\sin\varphi}{a+r\sin\varphi}u$$

$$- z\left[\frac{\sin\varphi}{(a+r\sin\varphi)^2}\frac{\partial w}{\partial \theta} - \frac{1}{(a+r\sin\varphi)^2}\frac{\partial^2 u}{\partial \theta^2}\right.$$

$$\left. + \frac{\cos\varphi}{r(a+r\sin\varphi)}v - \frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial u}{\partial \varphi}\right] \qquad (6)$$

$$\gamma_{\varphi\theta} = \frac{1}{r}\frac{\partial w}{\partial \varphi} - \frac{\cos\varphi}{a+r\sin\varphi}w + \frac{1}{a+r\sin\varphi}\frac{\partial v}{\partial \theta}$$

$$- z\left[\frac{2\cos\varphi}{r(a+r\sin\varphi)}w - \frac{4\sin\varphi\cos\varphi}{(a+r\sin\varphi)^2}w + \frac{1}{r(a+r\sin\varphi)}\frac{\partial v}{\partial \theta}\right.$$

$$\left. - \frac{2}{r(a+r\sin\varphi)}\frac{\partial^2 u}{\partial \varphi\partial \theta} + \frac{2\cos\varphi}{(a+r\sin\varphi)^2}\frac{\partial u}{\partial \theta} + \frac{2\sin\varphi}{r(a+r\sin\varphi)}\frac{\partial w}{\partial \varphi}\right]$$

From the basic assumption for thin shell theory, namely, that the stress components normal to the middle surface are small compared with the other stress components and may be neglected in the stress-strain relations, the following expressions are obtained for stresses in terms of strain and change of curvature:

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left[\epsilon_{\varphi_0} + \nu\epsilon_{\theta_0} - z(\chi_\varphi + \nu\chi_\theta) - (1+\nu)\alpha T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_{\theta_0} + \nu\epsilon_{\varphi_0} - z(\chi_\theta + \nu\chi_\varphi) - (1+\nu)\alpha T\right] \qquad (7)$$

$$\tau_{\varphi\theta} = \frac{E}{2(1+\nu)}\left(\gamma_{\varphi\theta_0} - 2z\chi_{\varphi\theta}\right)$$

Assuming all of the elastic constants are temperature dependent, the stress-strain relations of Eq. (7) may be written

$$\left.\begin{aligned} \sigma_\varphi &= E_1 \epsilon_{\varphi_0} + E_2 \epsilon_{\theta_0} - z E_1 \chi_\varphi - z E_2 \chi_\theta - F \\ \sigma_\theta &= E_1 \epsilon_{\theta_0} + E_2 \epsilon_{\varphi_0} - z E_1 \chi_\theta - z E_2 \chi_\varphi - F \\ \tau_{\varphi\theta} &= G \gamma_{\varphi\theta_0} - 2 G z \chi_{\varphi\theta} \end{aligned}\right\} \qquad (8)$$

where the temperature dependent quantities E_1, E_2 and F are defined by

$$\left.\begin{aligned} E_1 &\equiv \frac{E}{1-\nu^2} \\ E_2 &\equiv \frac{\nu E}{1-\nu^2} \\ F &= \frac{\alpha E T}{1-\nu} \end{aligned}\right\} \qquad (9)$$

The equations of equilibrium in toroidal coordinates are given by (Ref. 2)

$$\left.\begin{aligned} \frac{\partial \tau_{r\varphi}}{\partial z} + g_\varphi \tau_{r\varphi} + f_\varphi (z, \varphi, \theta ; r) &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial z} + g_\theta \tau_{r\theta} + f_\theta (z, \varphi, \theta ; r) &= 0 \end{aligned}\right\} \qquad (10)$$

where g_φ and g_θ are defined by

$$\left.\begin{aligned} g_\varphi &= \frac{2a + 3r \sin\varphi}{r(a + r\sin\varphi)} \\ g_\theta &= \frac{a + 3r\sin\varphi}{r(a + r\sin\varphi)} \end{aligned}\right\} \qquad (11)$$

and f_φ and f_θ are defined by

$$\left.\begin{aligned} f_\varphi &= \frac{1}{a+r\sin\varphi}\frac{\partial \tau_{\varphi\theta}}{\partial\theta} + \frac{1}{r}\frac{\partial \sigma_\varphi}{\partial\varphi} + \frac{\cos\varphi}{a+r\sin\varphi}\left(\sigma_\varphi - \sigma_\theta\right) \\ f_\theta &= \frac{1}{a+r\sin\varphi}\frac{\partial \sigma_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial \tau_{\varphi\theta}}{\partial\varphi} + \frac{2\cos\varphi}{a+r\sin\varphi}\tau_{\varphi\theta} \end{aligned}\right\} \quad (12)$$

Substituting Eqs. (8) in Eqs. (12), f_φ and f_θ may be written in terms of middle surface strains and changes of curvature according to

$$\left.\begin{aligned} f_\varphi = {} & \frac{1}{a+r\sin\varphi}\left[G\frac{\partial\gamma_{\varphi\theta_0}}{\partial\theta} + \gamma_{\varphi\theta_0}\frac{\partial G}{\partial\theta} - 2zG\frac{\partial\chi_{\varphi\theta}}{\partial\theta} - 2z\chi_{\varphi\theta}\frac{\partial G}{\partial\theta}\right] \\ & + \frac{1}{r}\left[E_1\frac{\partial\epsilon_{\varphi_0}}{\partial\varphi} + \epsilon_{\varphi_0}\frac{\partial E_1}{\partial\varphi} + E_2\frac{\partial\epsilon_{\theta_0}}{\partial\varphi} + \epsilon_{\theta_0}\frac{\partial E_2}{\partial\varphi} - zE_1\frac{\partial\chi_\varphi}{\partial\varphi}\right. \\ & \left. - z\chi_\varphi\frac{\partial E_1}{\partial\varphi} - zE_2\frac{\partial\chi_\theta}{\partial\varphi} - z\chi_\theta\frac{\partial E_2}{\partial\varphi} - \frac{\partial F}{\partial\varphi}\right] \\ & + \frac{\cos\varphi}{a+r\sin\varphi}\left[E_1\left(\epsilon_{\varphi_0}-\epsilon_{\theta_0}\right) + E_2\left(\epsilon_{\theta_0}-\epsilon_{\varphi_0}\right) - z\left(E_1-E_2\right)\left(\chi_\varphi-\chi_\theta\right)\right] \\ f_\theta = {} & \frac{1}{r}\left[G\frac{\partial\gamma_{\varphi\theta_0}}{\partial\varphi} + \gamma_{\varphi\theta_0}\frac{\partial G}{\partial\varphi} - 2zG\frac{\partial\chi_{\varphi\theta}}{\partial\varphi} - 2z\chi_{\varphi\theta}\frac{\partial G}{\partial\varphi}\right] \\ & + \frac{1}{a+r\sin\varphi}\left[E_1\frac{\partial\epsilon_{\theta_0}}{\partial\theta} + \epsilon_{\theta_0}\frac{\partial E_1}{\partial\theta} + E_2\frac{\partial\epsilon_{\varphi_0}}{\partial\theta} + \epsilon_{\varphi_0}\frac{\partial E_2}{\partial\theta} - zE_1\frac{\partial\chi_\theta}{\partial\theta}\right. \\ & \left. - z\chi_\theta\frac{\partial E_1}{\partial\theta} - zE_2\frac{\partial\chi_\varphi}{\partial\theta} - z\chi_\varphi\frac{\partial E_2}{\partial\theta} - \frac{\partial F}{\partial\theta}\right] \\ & + \frac{2\cos\varphi}{a+r\sin\varphi}\left[G\gamma_{\varphi\theta_0} - 2zG\chi_{\varphi\theta}\right] \end{aligned}\right\} \quad (13)$$

The solution of the first order differential equations, Eqs (10) for $\tau_{r\varphi}$ and $\tau_{r\theta}$ at the surfaces of the thin shell yields

$$\tau_{r\varphi}\Big|_{z_2} = e^{-g_\varphi(z_2-z_1)}\left\{-\int_{z_1}^{z_2} f_\varphi(z,\varphi,\theta;r)\, e^{g_\varphi(z-z_1)}\, dz + \tau_{r\varphi}\Big|_{z_1}\right\}$$

$$\tau_{r\theta}\Big|_{z_2} = e^{-g_\theta(z_2-z_1)}\left\{-\int_{z_1}^{z_2} f_\theta(z,\varphi,\theta;r)\, e^{g_\theta(z-z_1)}\, dz + \tau_{r\theta}\Big|_{z_1}\right\} \tag{14}$$

It is seen from Eqs. (11) that the exponents $g_\varphi(z_2-z_1)$ and $g_\theta(z_2-z_1)$ are of the order of h/r where h is the shell thickness and r is the principal radius of curvature. Hence the exponential terms are all approximately unity and Eqs. (14) reduce to

$$\left.\begin{aligned} \tau_{r\varphi}\Big|_2 - \tau_{r\varphi}\Big|_1 &= -\int_{z_1}^{z_2} f_\varphi(z,\varphi,\theta;r)\, dz \\ \tau_{r\theta}\Big|_2 - \tau_{r\theta}\Big|_1 &= -\int_{z_1}^{z_2} f_\theta(z,\varphi,\theta;r)\, dz \end{aligned}\right\} \tag{15}$$

which describe the discontinuity in shear stress across the thin shell. The integration of f_φ and f_θ in Eq. (13) is simply obtained since the middle surface strains and changes of curvature are independent of z. Defining the quantities

$$
\left.\begin{aligned}
D_0 &\equiv \int_{z_1}^{z_2} G(z)\,dz & D_4 &\equiv \int_{z_1}^{z_2} z\,E_1(z)\,dz \\
D_1 &\equiv \int_{z_1}^{z_2} E_1(z)\,dz & D_5 &\equiv \int_{z_1}^{z_2} z\,E_2(z)\,dz \\
D_2 &= \int_{z_1}^{z_2} E_2(z)\,dz & N_T &\equiv \int_{z_1}^{z_2} F(z)\,dz \\
D_3 &\equiv \int_{z_1}^{z_2} z\,G(z)\,dz
\end{aligned}\right\} \qquad (16)
$$

Eqs. (15) may be written

$$
\left.\begin{aligned}
\tau_{r\varphi}\big|_2 - \tau_{r\varphi}\big|_1 = &- \frac{1}{(a+r\sin\varphi)}\frac{\partial \gamma_{\varphi\theta_0}}{\partial\theta} - \frac{\gamma_{\varphi\theta_0}}{a+r\sin\varphi}\frac{\partial D_0}{\partial\theta} + \frac{2D_3}{a+r\sin\varphi}\frac{\partial \chi_{\varphi\theta}}{\partial\theta} \\
&+ \frac{2}{a+r\sin\varphi}\chi_{\varphi\theta}\frac{\partial D_3}{\partial\theta} - \frac{D_1}{r}\frac{\partial\epsilon_{\varphi_0}}{\partial\varphi} - \frac{\epsilon_{\varphi_0}}{r}\frac{\partial D_1}{\partial\varphi} - \frac{D_2}{r}\frac{\partial\epsilon_{\theta_0}}{\partial\varphi} \\
&- \frac{\epsilon_{\theta_0}}{r}\frac{\partial D_2}{\partial\varphi} + \frac{D_4}{r}\frac{\partial\chi_\varphi}{\partial\varphi} + \frac{1}{r}\chi_\varphi\frac{\partial D_4}{\partial\varphi} + \frac{D_5}{r}\frac{\partial\chi_\theta}{\partial\varphi} \\
&+ \frac{1}{r}\chi_\theta\frac{\partial D_5}{\partial\varphi} + \frac{1}{r}\frac{\partial N_T}{\partial\varphi} - \frac{(D_1-D_2)\cos\varphi}{a+r\sin\varphi}(\epsilon_{\varphi_0}-\epsilon_{\theta_0}) \\
&+ \frac{(D_4-D_5)\cos\varphi}{a+r\sin\varphi}(\chi_\varphi-\chi_\theta) \\
\tau_{r\theta}\big|_2 - \tau_{r\theta}\big|_1 = &- \frac{D_0}{r}\frac{\partial\gamma_{\varphi\theta_0}}{\partial\varphi} - \frac{\gamma_{\varphi\theta_0}}{r}\frac{\partial D_0}{\partial\varphi} + \frac{2D_3}{r}\frac{\partial\chi_{\varphi\theta}}{\partial\varphi} + \frac{2}{r}\chi_{\varphi\theta}\frac{\partial D_3}{\partial\varphi} \\
&- \frac{D_1}{a+r\sin\varphi}\frac{\partial\epsilon_{\theta_0}}{\partial\theta} - \frac{\epsilon_{\theta_0}}{a+r\sin\varphi}\frac{\partial D_1}{\partial\theta} - \frac{D_2}{a+r\sin\varphi}\frac{\partial\epsilon_{\varphi_0}}{\partial\theta} \\
&- \frac{\epsilon_{\varphi_0}}{a+r\sin\varphi}\frac{\partial D_2}{\partial\theta} + \frac{D_4}{a+r\sin\varphi}\frac{\partial\chi_\theta}{\partial\theta} + \frac{\chi_\theta}{a+r\sin\varphi}\frac{\partial D_4}{\partial\theta} \\
&+ \frac{D_5}{a+r\sin\varphi}\frac{\partial\chi_\varphi}{\partial\theta} + \frac{\chi_\varphi}{a+r\sin\varphi}\frac{\partial D_5}{\partial\theta} + \frac{1}{a+r\sin\varphi}\frac{\partial N_T}{\partial\theta}
\end{aligned}\right\} \qquad (17)
$$

Eqs. (17) may be written in terms of displacements by substituting for middle surface strains and changes of curvature the expressions of Eqs. (4). The derivatives of these expressions which appear in Eqs. (17) are as follows:

$$\frac{1}{r}\frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} = \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial u}{\partial \varphi}$$

$$\frac{1}{r}\frac{\partial \epsilon_{\theta_0}}{\partial \varphi} = \frac{1}{(a+r\sin\varphi)}\Big[\frac{1}{r}\frac{\partial^2 w}{\partial\varphi\partial\theta} - \frac{\cos\varphi}{(a+r\sin\varphi)}\frac{\partial w}{\partial\theta} + \frac{\cos\varphi}{r}\frac{\partial v}{\partial\varphi}$$

$$- \frac{\sin\varphi}{r}v - \frac{\cos^2\varphi}{a+r\sin\varphi}v + \frac{\sin\varphi}{r}\frac{\partial u}{\partial\varphi} + \frac{\cos\varphi}{r}u$$

$$- \frac{\sin\varphi\cos\varphi}{a+r\sin\varphi}u\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial \epsilon_{\varphi_0}}{\partial\theta} = \frac{1}{r(a+r\sin\varphi)}\Big[\frac{\partial^2 v}{\partial\varphi\partial\theta} + \frac{\partial u}{\partial\theta}\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial \epsilon_{\theta_0}}{\partial\theta} = \frac{1}{(a+r\sin\varphi)^2}\Big[\frac{\partial^2 w}{\partial\theta} + \cos\varphi\frac{\partial v}{\partial\theta} + \sin\varphi\frac{\partial u}{\partial\theta}\Big]$$

$$\frac{1}{r}\frac{\partial \gamma_{\varphi\theta_0}}{\partial\varphi} = \frac{1}{r(a+r\sin\varphi)}\Big[\frac{a+r\sin\varphi}{r}\frac{\partial^2 w}{\partial\varphi^2} - \cos\varphi\frac{\partial w}{\partial\varphi} + \sin\varphi\cdot w$$

$$+ \frac{r\cos^2\varphi}{a+r\sin\varphi}w + \frac{\partial^2 v}{\partial\varphi\partial\theta} - \frac{r\cos\varphi}{a+r\sin\varphi}\frac{\partial v}{\partial\theta}\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial \gamma_{\varphi\theta_0}}{\partial\theta} = \frac{1}{(a+r\sin\varphi)^2}\Big[\frac{a+r\sin\varphi}{r}\frac{\partial^2 w}{\partial\varphi\partial\theta} - \cos\varphi\frac{\partial w}{\partial\theta} + \frac{\partial v}{\partial\theta}\Big]$$

$$\frac{1}{r}\frac{\partial \chi_\varphi}{\partial\varphi} = \frac{1}{r^3}\Big[\frac{\partial^2 v}{\partial\varphi^2} - \frac{\partial^3 u}{\partial\varphi^3}\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial \chi_\varphi}{\partial\theta} = \frac{1}{r^2(a+r\sin\varphi)}\Big[\frac{\partial^2 v}{\partial\varphi\partial\theta} - \frac{\partial^3 u}{\partial\varphi^2\partial\theta}\Big] \tag{18}$$

$$\frac{1}{r}\frac{\partial \chi_\theta}{\partial \varphi} = \frac{1}{r(a+r\sin\varphi)^2}\Big[\sin\varphi\frac{\partial^2 w}{\partial\varphi\partial\theta} + \cos\varphi\frac{\partial w}{\partial\theta} - \frac{2r\sin\varphi\cos\varphi}{a+r\sin\varphi}\frac{\partial w}{\partial\theta}$$

$$- \frac{\partial^3 u}{\partial\varphi\partial\theta^2} + \frac{2r\cos\varphi}{a+r\sin\varphi}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\varphi(a+r\sin\varphi)}{r}\frac{\partial v}{\partial\varphi}$$

$$- \frac{\sin\varphi(a+r\sin\varphi)}{r}v - \cos^2\varphi\cdot v - \frac{\cos\varphi(a+r\sin\varphi)}{r}\frac{\partial^2 u}{\partial\varphi^2}$$

$$+ \frac{\sin\varphi(a+r\sin\varphi)}{r}\frac{\partial u}{\partial\varphi} + \cos^2\varphi\frac{\partial u}{\partial\varphi}\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial\chi_\theta}{\partial\theta} = \frac{1}{(a+r\sin\varphi)^2}\Big[\frac{\sin\varphi}{a+r\sin\varphi}\frac{\partial^2 w}{\partial\theta^2} - \frac{1}{a+r\sin\varphi}\frac{\partial^3 u}{\partial\theta^3}$$

$$+ \frac{\cos\varphi}{r}\frac{\partial v}{\partial\theta} - \frac{\cos\varphi}{r}\frac{\partial^2 u}{\partial\varphi\partial\theta}\Big]$$

$$\frac{1}{r}\frac{\partial\chi_{\varphi\theta}}{\partial\varphi} = \frac{1}{2r(a+r\sin\varphi)}\Big[\frac{2\cos\varphi}{r}\frac{\partial w}{\partial\varphi} - \frac{2\sin\varphi}{r}w - \frac{2\cos^2\varphi}{a+r\sin\varphi}w$$

$$- \frac{4\sin\varphi\cos\varphi}{a+r\sin\varphi}\frac{\partial w}{\partial\varphi} - \frac{4(\cos^2\varphi-\sin^2\varphi)}{a+r\sin\varphi}w - \frac{8r\sin\varphi\cos^2\varphi}{(a+r\sin\varphi)^2}w$$

$$+ \frac{1}{r}\frac{\partial^2 v}{\partial\varphi\partial\theta} - \frac{\cos\varphi}{a+r\sin\varphi}\frac{\partial v}{\partial\theta} - \frac{2}{r}\frac{\partial^3 u}{\partial\varphi^2\partial\theta} + \frac{2\cos\varphi}{(a+r\sin\varphi)}\frac{\partial^2 u}{\partial\varphi\partial\theta}\Big]$$

$$\frac{1}{a+r\sin\varphi}\frac{\partial\chi_{\varphi\theta}}{\partial\theta} = \frac{1}{2(a+r\sin\varphi)^2}\Big[\frac{2\cos\varphi}{r}\frac{\partial w}{\partial\theta} - \frac{4\sin\varphi\cos\varphi}{a+r\sin\varphi}\frac{\partial w}{\partial\theta}$$

$$+ \frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} - \frac{2}{r}\frac{\partial^3 u}{\partial\varphi\partial\theta^2} + \frac{2\cos\varphi}{a+r\sin\varphi}\frac{\partial^2 u}{\partial\theta^2}$$

$$+ \frac{2\sin\varphi}{r}\frac{\partial^2 w}{\partial\varphi\partial\theta}\Big] \qquad (18)$$

For the case in which Poisson's ratio is not temperature dependent, the integrals D_3, D_4 and D_5 of Eqs. (16) reduce to a constant times the integral of $zE(z)$, which is generally defined such that

$$\int_{z_1}^{z_2} z\,E(z)\,dz = 0 \quad . \tag{19}$$

This expression describes the neutral surface ($z=0$) of the shell. Since Poisson's ratio is temperature dependent in the case being considered, this simplification is not possible and only one of the integrals D_3, D_4 or D_5 may be arbitrarily set equal to zero, which defines the neutral surface in this case.

In addition to Eqs. (17) which describe the shear stress discontinuities across the thin shell, an equation can be obtained for the discontinuity of normal stress across the shell. A force balance normal to the plane of the shell, expressed in general curvilinear coordinates (Ref. 1), requires that

$$\frac{\partial \alpha_3 Q_2}{\partial \xi_2} + \frac{\partial \alpha_2 Q_3}{\partial \xi_3} + N_2 \frac{\alpha_2 \alpha_3}{R_2} + N_3 \frac{\alpha_2 \alpha_3}{R_3} + \alpha_2 \alpha_3 P_z = 0 \ , \tag{20}$$

where the subscripts 2 and 3 refer to φ and θ, respectively, $\xi_2 = \varphi$, $\xi_3 = \theta$ and the quantities α_2, α_3, R_2 and R_3 have been previously defined. Similarly, a balance of moments about the φ and θ axes gives

$$\frac{\partial \alpha_3 M_{23}}{\partial \xi_2} - \frac{\partial \alpha_2 M_3}{\partial \xi_3} + M_2 \frac{\partial \alpha_2}{\partial \xi_3} + M_{32} \frac{\partial \alpha_3}{\partial \xi_2} + \alpha_2 \alpha_3 Q_3 = 0 \tag{21}$$

and

$$\frac{\partial \alpha_2 M_{32}}{\partial \xi_3} - \frac{\partial \alpha_3 M_2}{\partial \xi_2} + M_3 \frac{\partial \alpha_3}{\partial \xi_2} + M_{23} \frac{\partial \alpha_2}{\partial \xi_3} + \alpha_2 \alpha_3 Q_2 = 0 \tag{22}.$$

Rewriting Eqs. (20) – (22) in terms of the coordinates φ and θ using the respective values for α_2, α_3, R_2 and R_3, there is obtained

$$\frac{1}{r}\frac{\partial Q_\varphi}{\partial\varphi} + \frac{\cos\varphi}{a+r\sin\varphi}Q_\varphi + \frac{1}{a+r\sin\varphi}\frac{\partial Q_\theta}{\partial\theta} + \frac{1}{r}N_\varphi + \frac{\sin\varphi}{a+r\sin\varphi}N_\theta + P_z = 0 \tag{23}$$

$$-\frac{1}{r}\frac{\partial M_{\varphi\theta}}{\partial\varphi} + \frac{1}{a+r\sin\varphi}\frac{\partial M_\theta}{\partial\theta} - \frac{2\cos\varphi}{a+r\sin\varphi}M_{\varphi\theta} - Q_\theta = 0 \tag{24}$$

$$r\frac{\partial M_{\varphi\theta}}{\partial\theta} - (a+r\sin\varphi)\frac{\partial M_\varphi}{\partial\varphi} - M_\varphi r\cos\varphi + M_\theta r\cos\varphi + Q_\varphi r(a+r\sin\varphi) = 0 \tag{25}$$

Substituting Q_φ and Q_θ from Eqs. (24) and (25) in Eq. (23) there is obtained one equation in terms of the sectional forces and moments; namely

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial^2 M_\varphi}{\partial\varphi^2} + \frac{1}{(a+r\sin\varphi)^2}\frac{\partial^2 M_\theta}{\partial\theta^2} - \frac{2}{r(a+r\sin\varphi)}\frac{\partial^2 M_{\varphi\theta}}{\partial\varphi\,\partial\theta} + \frac{2\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial M_\varphi}{\partial\varphi}\\ &- \frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial M_\theta}{\partial\varphi} - \frac{2\cos\varphi}{(a+r\sin\varphi)^2}\frac{\partial M_{\varphi\theta}}{\partial\theta} - \frac{\sin\varphi}{r(a+r\sin\varphi)}M_\varphi\\ &+ \frac{\sin\varphi}{r(a+r\sin\varphi)}M_\theta + \frac{1}{r}N_\varphi + \frac{\sin\varphi}{a+r\sin\varphi}N_\theta + P_z = 0\end{aligned} \tag{26}$$

The sectional forces and moments are defined as

$$\left.\begin{aligned}N_\varphi &= \int_{z_1}^{z_2}\sigma_\varphi\left(1-\frac{z\sin\varphi}{a+r\sin\varphi}\right)dz, \quad N_\theta = \int_{z_1}^{z_2}\sigma_\theta\left(1-\frac{z}{r}\right)dz\\ M_\varphi &= \int_{z_1}^{z_2}\sigma_\varphi\left(1-\frac{z\sin\varphi}{a+r\sin\varphi}\right)z\,dz, \quad M_\theta = \int_{z_1}^{z_2}\sigma_\theta\left(1-\frac{z}{r}\right)z\,dz\\ M_{\varphi\theta} &= \int_{z_1}^{z_2}\tau_{\varphi\theta}\left(1-\frac{z\sin\varphi}{a+r\sin\varphi}\right)z\,dz\end{aligned}\right\} \tag{27}$$

The second term in each of the parentheses of Eq. (27) is proportional to the shell thickness divided by the radius of curvature and can be ignored for all cases being considered. Hence Eqs. (27) reduce to

$$\left.\begin{aligned} N_\varphi &= \int_{z_1}^{z_2} \sigma_\varphi \, dz \;, \qquad N_\theta = \int_{z_1}^{z_2} \sigma_\theta \, dz \\ M_\varphi &= \int_{z_1}^{z_2} \sigma_\varphi \, z\,dz \;, \qquad M_\theta = \int_{z_1}^{z_2} \sigma_\theta \, z\,dz \\ M_{\varphi\theta} &= \int_{z_1}^{z_2} \tau_{\varphi\theta} \, z\,dz \end{aligned}\right\} \qquad (28)$$

Substituting the stresses of Eq. (8) in Eqs. (28) and integrating, there is obtained

$$\left.\begin{aligned} N_\varphi &= D_1 \epsilon_{\varphi_0} + D_2 \epsilon_{\theta_0} - D_4 \chi_\varphi - D_5 \chi_\theta - N_T \\ N_\theta &= D_1 \epsilon_{\theta_0} + D_2 \epsilon_{\varphi_0} - D_4 \chi_\theta - D_5 \chi_\varphi - N_T \\ M_\varphi &= D_4 \epsilon_{\varphi_0} + D_5 \epsilon_{\theta_0} - D_6 \chi_\varphi - D_7 \chi_\theta - M_T \\ M_\theta &= D_4 \epsilon_{\theta_0} + D_5 \epsilon_{\varphi_0} - D_6 \chi_\theta - D_7 \chi_\varphi - M_T \\ M_{\varphi\theta} &= D_3 \gamma_{\varphi\theta_0} - 2 D_8 \chi_{\varphi\theta} \end{aligned}\right\} \qquad (29)$$

where, in addition to the integrals, Eq. (16), of the elastic constants defined in Eqs. (9), there are defined the additional integrals

$$
\left.
\begin{aligned}
D_6 &= \int_{z_1}^{z_2} z^2 E_1(z)\,dz & \qquad D_8 &= \int_{z_1}^{z_2} z^2 G(z)\,dz \\
D_7 &= \int_{z_1}^{z_2} z^2 E_2(z)\,dz & \qquad M_T &= \int_{z_1}^{z_2} z F(z)\,dz
\end{aligned}
\right\} \quad (30)
$$

The derivatives of the sectional forces and moments appearing in Eq. (26), including temperature dependence of the elastic constants, are given by the following:

$$
\left.
\begin{aligned}
\frac{\partial M_\varphi}{\partial \varphi} &= D_4 \frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} + \epsilon_{\varphi_0} \frac{\partial D_4}{\partial \varphi} + D_5 \frac{\partial \epsilon_{\theta_0}}{\partial \varphi} + \epsilon_{\theta_0} \frac{\partial D_5}{\partial \varphi} - D_6 \frac{\partial \chi_\varphi}{\partial \varphi} \\
&\quad - \chi_\varphi \frac{\partial D_6}{\partial \varphi} - D_7 \frac{\partial \chi_\theta}{\partial \varphi} - \chi_\theta \frac{\partial D_7}{\partial \varphi} - \frac{\partial M_T}{\partial \varphi} \\
\frac{\partial M_\theta}{\partial \varphi} &= D_4 \frac{\partial \epsilon_{\theta_0}}{\partial \varphi} + \epsilon_{\theta_0} \frac{\partial D_4}{\partial \varphi} + D_5 \frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} + \epsilon_{\varphi_0} \frac{\partial D_5}{\partial \varphi} - D_6 \frac{\partial \chi_\theta}{\partial \varphi} \\
&\quad - \chi_\theta \frac{\partial D_6}{\partial \varphi} - D_7 \frac{\partial \chi_\varphi}{\partial \varphi} - \chi_\varphi \frac{\partial D_7}{\partial \varphi} - \frac{\partial M_T}{\partial \varphi} \\
\frac{\partial M_{\varphi\theta}}{\partial \theta} &= D_3 \frac{\partial \gamma_{\varphi\theta_0}}{\partial \theta} + \gamma_{\varphi\theta_0} \frac{\partial D_3}{\partial \theta} - 2 D_8 \frac{\partial \chi_{\varphi\theta}}{\partial \theta} - 2 \chi_{\varphi\theta} \frac{\partial D_8}{\partial \theta} \\
\frac{\partial^2 M_\varphi}{\partial \varphi^2} &= D_4 \frac{\partial^2 \epsilon_{\varphi_0}}{\partial \varphi^2} + 2 \frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} \frac{\partial D_4}{\partial \varphi} + \epsilon_{\varphi_0} \frac{\partial^2 D_4}{\partial \varphi^2} + D_5 \frac{\partial^2 \epsilon_{\theta_0}}{\partial \varphi^2} \\
&\quad + 2 \frac{\partial \epsilon_{\theta_0}}{\partial \varphi} \frac{\partial D_5}{\partial \varphi} + \epsilon_{\theta_0} \frac{\partial^2 D_5}{\partial \varphi^2} - D_6 \frac{\partial^2 \chi_\varphi}{\partial \varphi^2} - 2 \frac{\partial D_6}{\partial \varphi} \frac{\partial \chi_\varphi}{\partial \varphi} \\
&\quad - \chi_\varphi \frac{\partial^2 D_6}{\partial \varphi^2} - D_7 \frac{\partial^2 \chi_\theta}{\partial \varphi^2} - 2 \frac{\partial D_7}{\partial \varphi} \frac{\partial \chi_\theta}{\partial \varphi} - \chi_\theta \frac{\partial^2 D_7}{\partial \varphi^2} \\
&\quad - \frac{\partial^2 M_T}{\partial \varphi^2}
\end{aligned}
\right\} \quad (31)
$$

$$
\left.
\begin{aligned}
\frac{\partial^2 M_\theta}{\partial\theta^2} = {} & D_4 \frac{\partial^2 \epsilon_{\theta_0}}{\partial\theta^2} + 2\frac{\partial D_4}{\partial\theta}\frac{\partial\epsilon_{\theta_0}}{\partial\theta} + \epsilon_{\theta_0}\frac{\partial^2 D_4}{\partial\theta^2} + D_5\frac{\partial^2\epsilon_{\varphi_0}}{\partial\theta^2} \\
& + 2\frac{\partial D_5}{\partial\theta}\frac{\partial\epsilon_{\varphi_0}}{\partial\theta} + \epsilon_{\varphi_0}\frac{\partial^2 D_5}{\partial\theta^2} - D_6\frac{\partial^2 X_\theta}{\partial\theta^2} - 2\frac{\partial D_6}{\partial\theta}\frac{\partial X_\theta}{\partial\theta} \\
& - X_\theta\frac{\partial^2 D_6}{\partial\theta^2} - D_7\frac{\partial^2 X_\varphi}{\partial\theta^2} - 2\frac{\partial D_7}{\partial\theta}\frac{\partial X_\varphi}{\partial\theta} - X_\varphi\frac{\partial^2 D_7}{\partial\theta^2} - \frac{\partial^2 M_T}{\partial\theta^2} \\
\frac{\partial^2 M_{\varphi\theta}}{\partial\varphi\,\partial\theta} = {} & D_3\frac{\partial^2\gamma_{\varphi\theta_0}}{\partial\varphi\,\partial\theta} + \frac{\partial D_3}{\partial\varphi}\frac{\partial\gamma_{\varphi\theta_0}}{\partial\theta} + \frac{\partial\gamma_{\varphi\theta_0}}{\partial\varphi}\frac{\partial D_3}{\partial\theta} + \gamma_{\varphi\theta_0}\frac{\partial^2 D_3}{\partial\varphi\,\partial\theta} \\
& - 2D_8\frac{\partial^2 X_{\varphi\theta}}{\partial\varphi\,\partial\theta} - 2\frac{\partial D_8}{\partial\varphi}\frac{\partial X_{\varphi\theta}}{\partial\theta} - 2\frac{\partial D_8}{\partial\theta}\frac{\partial X_{\varphi\theta}}{\partial\varphi} - 2X_{\varphi\theta}\frac{\partial^2 D_8}{\partial\varphi\,\partial\theta}
\end{aligned}
\right\} \quad (31)
$$

Eq. (26) may be written in terms of displacements by substituting for the middle surface strains and changes of curvature in Eqs. (31), the expressions previously derived in terms of displacements. These expressions are given in Eq. (4) and their first derivatives with respect to the coordinates φ and θ are given in Eqs. (18). Second derivatives which also appear in Eqs. (31) are given by the following:

$$
\left.
\begin{aligned}
\frac{\partial^2\epsilon_{\varphi_0}}{\partial\varphi^2} = {} & \frac{1}{r}\frac{\partial^3 v}{\partial\varphi^3} + \frac{1}{r}\frac{\partial^2 u}{\partial\varphi^2} \\
\frac{\partial^2\epsilon_{\varphi_0}}{\partial\theta^2} = {} & \frac{1}{r}\left[\frac{\partial^3 v}{\partial\varphi\,\partial\theta^2} + \frac{\partial^2 u}{\partial\theta^2}\right] \\
\frac{\partial^2\epsilon_{\theta_0}}{\partial\varphi^2} = {} & \frac{1}{a+r\sin\varphi}\Bigg[\frac{\partial^3 w}{\partial\varphi^2\,\partial\theta} - \frac{2r\cos\varphi}{a+r\sin\varphi}\frac{\partial^2 w}{\partial\varphi\,\partial\theta} \\
& + \frac{r^2(1+\cos^2\varphi)+ar\sin\varphi}{(a+r\sin\varphi)^2}\frac{\partial w}{\partial\theta} + \cos\varphi\frac{\partial^2 v}{\partial\varphi^2} + \sin\varphi\frac{\partial^2 u}{\partial\varphi^2} \\
& - \frac{2(a\sin\varphi+r)}{a+r\sin\varphi}\frac{\partial v}{\partial\varphi} + \frac{2a\cos\varphi}{a+r\sin\varphi}\frac{\partial u}{\partial\varphi} \\
& + \frac{\cos\varphi\,(ar\sin\varphi+2r^2-a^2)}{(a+r\sin\varphi)^2}\,v - \frac{ar(1+\cos^2\varphi)+a^2\sin\varphi}{(a+r\sin\varphi)^2}\,u\Bigg]
\end{aligned}
\right\} \quad (32)
$$

$$\frac{\partial^2 \epsilon_{\theta_0}}{\partial \theta^2} = \frac{1}{a + r\sin\varphi}\left[\frac{\partial^3 w}{\partial \theta^3} + \cos\varphi \frac{\partial^2 v}{\partial \theta^2} + \sin\varphi \frac{\partial^2 u}{\partial \theta^2}\right]$$

$$\frac{\partial^2 \gamma_{\varphi\theta_0}}{\partial\varphi\,\partial\theta} = \frac{1}{a + r\sin\varphi}\left[\frac{a + r\sin\varphi}{r}\frac{\partial^3 w}{\partial\varphi^2\partial\theta} - \cos\varphi\frac{\partial^2 w}{\partial\varphi\partial\theta} + \sin\varphi\frac{\partial w}{\partial\theta}\right.$$

$$\left. + \frac{r\cos^2\varphi}{a + r\sin\varphi}\frac{\partial w}{\partial\theta} + \frac{\partial^3 v}{\partial\varphi\partial\theta^2} - \frac{r\cos\varphi}{a + r\sin\varphi}\frac{\partial^2 v}{\partial\theta^2}\right]$$

$$\frac{\partial^2 \chi_\varphi}{\partial\varphi^2} = \frac{1}{r^2}\left[\frac{\partial^3 v}{\partial\varphi^3} - \frac{\partial^4 u}{\partial\varphi^4}\right]$$

$$\frac{\partial^2 \chi_\varphi}{\partial\theta^2} = \frac{1}{r^2}\left[\frac{\partial^3 v}{\partial\varphi\partial\theta^2} - \frac{\partial^4 u}{\partial\varphi^2\partial\theta^2}\right]$$

$$\frac{\partial^2 \chi_\theta}{\partial\varphi^2} = \frac{\sin\varphi}{(a + r\sin\varphi)^2}\frac{\partial^3 w}{\partial\varphi^2\partial\theta} + \frac{2\cos\varphi\,(a - r\sin\varphi)}{(a + r\sin\varphi)^3}\frac{\partial^2 w}{\partial\varphi\partial\theta}$$

$$- \frac{\partial w}{\partial\theta}\left[\frac{(a^2 - r^2)\sin\varphi + r\cos^2\varphi\,(4a - r\sin\varphi)}{(a + r\sin\varphi)^4}\right]$$

$$- \frac{1}{(a + r\sin\varphi)^2}\frac{\partial^4 u}{\partial\varphi^2\partial\theta^2} + \frac{4r\cos\varphi}{(a + r\sin\varphi)^3}\frac{\partial^3 u}{\partial\varphi\partial\theta^2} - \frac{\cos\varphi}{r(a + r\sin\varphi)}\frac{\partial^3 u}{\partial\varphi^3}$$

$$+ \frac{2(r + a\sin\varphi)}{r(a + r\sin\varphi)^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\partial^2 u}{\partial\theta^2}\left[\frac{2r^2(2\cos^2\varphi + 1) + 2ar\sin\varphi}{(a + r\sin\varphi)^4}\right]$$

$$+ \frac{\cos\varphi}{r(a + r\sin\varphi)}\frac{\partial^2 v}{\partial\varphi^2} + \frac{\partial u}{\partial\varphi}\left[\frac{(a^2 - 2r^2\cos^2\varphi)\cos\varphi - r(a + 2r\sin\varphi)\sin\varphi\cos\varphi}{r(a + r\sin\varphi)^3}\right]$$

$$- \frac{2(r + a\sin\varphi)}{r(a + r\sin\varphi)^2}\frac{\partial v}{\partial\varphi} + v\left[\frac{(a + 2r)\,r\sin\varphi\cos\varphi - (a^2 - 2r^2\cos^2\varphi)\cos\varphi}{r(a + r\sin\varphi)^3}\right] \qquad (32)$$

$$\left.\begin{aligned}
\frac{\partial^2 \chi_\theta}{\partial \theta^2} &= \frac{1}{a + r\sin\varphi}\left[\frac{\sin\varphi}{a + r\sin\varphi}\frac{\partial^3 w}{\partial \theta^3} - \frac{1}{a + r\sin\varphi}\frac{\partial^4 u}{\partial \theta^4}\right. \\
&\qquad \left. + \frac{\cos\varphi}{r}\frac{\partial^2 v}{\partial \theta^2} - \frac{\cos\varphi}{r}\frac{\partial^3 u}{\partial\varphi\,\partial\theta^2}\right] \\
\frac{\partial^2 \chi_{\varphi\theta}}{\partial\varphi\,\partial\theta} &= \frac{1}{a + r\sin\varphi}\left[\frac{\cos\varphi\,(a - r\sin\varphi)}{r(a + r\sin\varphi)}\frac{\partial^2 w}{\partial\varphi\,\partial\theta}\right. \\
&\qquad - \frac{a^2\sin\varphi + 3ar\cos^2\varphi + 7r^2\sin\varphi\cos^2\varphi - r^2\sin^3\varphi}{r(a + r\sin\varphi)^2}\frac{\partial w}{\partial\theta} \\
&\qquad \left. + \frac{1}{2r}\frac{\partial^3 v}{\partial\varphi\,\partial\theta^2} - \frac{\cos\varphi}{2(a + r\sin\varphi)}\frac{\partial^2 v}{\partial\theta^2} - \frac{1}{r}\frac{\partial^4 u}{\partial\varphi^2\,\partial\theta^2} + \frac{\cos\varphi}{a + r\sin\varphi}\frac{\partial^3 u}{\partial\varphi\,\partial\theta^2}\right]
\end{aligned}\right\} \quad (32)$$

Eqs. (17) and (26) constitute three equations in the three displacement components (u, v, w) in terms of shear and normal surface loads applied to the thin shell. From continuity of the three components of surface stress ($\tau_{r\varphi}$, $\tau_{r\theta}$, σ_z) at the two surfaces of the shell, the stresses can be related to the displacement components in the two adjoining media to yield a total of nine simultaneous equations in the three neutral surface displacements plus the three displacements at the surface of each of the adjoining media.

The problem may be looked at alternatively by considering that the two media adjoining the thin shell are subject to surface stress boundary conditions; i.e., the "known" surface stresses at each boundary are related to displacements at the surface through the stress-strain relations, using the appropriate elastic constants for each region. Since the actual boundary stresses are not known, but rather the difference in each stress component across the thin shell is known in terms of the neutral surface displacements, e.g., $\tau_{r\varphi}|_1 - \tau_{r\varphi}|_2 = f(u, v, w)$, etc., the boundary conditions become three equations in each of the three stress components. These equations reduce to nine equations in nine displacement components (three for the thin shell neutral surface and three at the surface of each of the adjoining media) when the stresses are eliminated.

Since spherical coordinates are a limiting case of toroidal coordinates, it is apparent that the above development applies to the spherical thin shell in the limit as the radius a approaches zero. No singularity is introduced in passing to this limit since the radius a appears only in conjunction with the term $r\sin\varphi$, e.g., $(a + r\sin\varphi)$.

<u>Reduction to Axially Symmetric Case</u>

The conditions for axial symmetry require that

$$w(r, \varphi, \theta) = \frac{\partial f}{\partial \theta} = 0 \quad , \tag{33}$$

where w is the azimuthal component of the displacement vector in the θ-direction and f is any function of the coordinates (r, φ, θ). It will be shown that these conditions result in two equations in the non-vanishing displacement components u and v in terms of the stress discontinuities $\tau_{r\varphi}|_2 - \tau_{r\varphi}|_1$ and $\sigma_z|_2 - \sigma_z|_1$ across the thin shell. The stress difference $\tau_{r\theta}|_2 - \tau_{r\theta}|_1$ becomes zero in the axially symmetric case.

In view of the conditions of Eq. (33), the neutral surface strains and changes of curvature, from Eq. (4), reduce to

$$
\left.
\begin{aligned}
\epsilon_{\varphi_0} &= \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r} \\
\epsilon_{\theta_0} &= \frac{\cos\varphi}{a + r\sin\varphi}\cdot v + \frac{\sin\varphi}{a + r\sin\varphi}\cdot u \\
\gamma_{\varphi\theta_0} &= 0 \\
\chi_\varphi &= \frac{1}{r^2}\left(\frac{\partial v}{\partial \varphi} - \frac{\partial^2 u}{\partial \varphi^2}\right) \\
\chi_\theta &= \frac{\cos\varphi}{r(a + r\sin\varphi)}\left(v - \frac{\partial u}{\partial \varphi}\right) \\
\chi_{\varphi\theta} &= 0
\end{aligned}
\right\} \tag{34}
$$

The shear stress discontinuities of Eq. (17) then reduce to

$$
\left.
\begin{aligned}
\tau_{r\varphi}\big|_2 - \tau_{r\varphi}\big|_1 = &- \frac{D_1}{r}\frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} - \frac{\epsilon_{\varphi_0}}{r}\frac{\partial D_1}{\partial \varphi} - \frac{D_2}{r}\frac{\partial \epsilon_{\theta_0}}{\partial \varphi} \\
&- \frac{\epsilon_{\theta_0}}{r}\frac{\partial D_2}{\partial \varphi} + \frac{D_4}{r}\frac{\partial \chi_\varphi}{\partial \varphi} + \frac{1}{r}\chi_\varphi \frac{\partial D_4}{\partial \varphi} \\
&+ \frac{D_5}{r}\frac{\partial \chi_\theta}{\partial \varphi} + \frac{1}{r}\chi_\theta \frac{\partial D_5}{\partial \varphi} + \frac{1}{r}\frac{\partial N_T}{\partial \varphi} \\
&- \frac{(D_1 - D_2)\cos\varphi}{a + r\sin\varphi}\left(\epsilon_{\varphi_0} - \epsilon_{\theta_0}\right) \\
\tau_{r\theta}\big|_2 - \tau_{r\theta}\big|_1 = &\; 0
\end{aligned}
\right\} \tag{35}
$$

The derivatives which appear in Eq. (35), from the non-axially symmetric case reported in Eq. (18), reduce to

$$
\left.
\begin{aligned}
\frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} &= \frac{1}{r}\left(\frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial u}{\partial \varphi}\right) \\
\frac{\partial \epsilon_{\theta_0}}{\partial \varphi} &= \frac{1}{(a + r\sin\varphi)}\Big[\cos\varphi \frac{\partial v}{\partial \varphi} - v\sin\varphi - \frac{r\cos^2\varphi}{a + r\sin\varphi}\, v \\
&\qquad + \sin\varphi \frac{\partial u}{\partial \varphi} + u\cos\varphi - \frac{r\sin\varphi\cos\varphi}{a + r\sin\varphi}\, u\Big] \\
\frac{\partial \epsilon_{\varphi_0}}{\partial \theta} &= 0 \\
\frac{\partial \epsilon_{\theta_0}}{\partial \theta} &= 0 \\
\frac{\partial \gamma_{\varphi\theta_0}}{\partial \varphi} &= 0 \\
\frac{\partial \gamma_{\varphi\theta_0}}{\partial \theta} &= 0 \\
\frac{\partial \chi_\varphi}{\partial \varphi} &= \frac{1}{r^2}\left(\frac{\partial^2 v}{\partial \varphi^2} - \frac{\partial^3 u}{\partial \varphi^3}\right) \\
\frac{\partial \chi_\varphi}{\partial \theta} &= 0 \\
\frac{\partial \chi_\theta}{\partial \varphi} &= \frac{1}{(a + r\sin\varphi)}\Big[\frac{\cos\varphi}{r}\frac{\partial v}{\partial \varphi} - \frac{\sin\varphi}{r}\, v - \frac{v\cos^2\varphi}{(a + r\sin\varphi)} \\
&\qquad - \frac{\cos\varphi}{r}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\sin\varphi}{r}\frac{\partial u}{\partial \varphi} + \frac{\cos^2\varphi}{(a + r\sin\varphi)}\frac{\partial u}{\partial \varphi}\Big] \\
\frac{\partial \chi_\theta}{\partial \theta} &= 0 \\
\frac{\partial \chi_{\varphi\theta}}{\partial \varphi} &= 0 \\
\frac{\partial \chi_{\varphi\theta}}{\partial \theta} &= 0
\end{aligned}
\right\} \qquad (36)
$$

Finally, Eq. (26), which defines the normal stress discontinuity across the thin shell, or the pressure loading P_z, where

$$P_z \equiv \sigma_z|_2 - \sigma_z|_1 \,,$$

becomes

$$\frac{1}{r^2}\frac{\partial^2 M_\varphi}{\partial \varphi^2} + \frac{2\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial M_\varphi}{\partial \varphi} - \frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial M_\theta}{\partial \varphi}$$

$$- \frac{\sin\varphi}{r(a+r\sin\varphi)} M_\varphi + \frac{\sin\varphi}{r(a+r\sin\varphi)} M_\theta + \frac{1}{r} N_\varphi \qquad (37)$$

$$+ \frac{\sin\varphi}{a+r\sin\varphi} N_\theta + P_z = 0 \,,$$

where the sectional forces and moments are defined in Eq. (28), in terms of the stress components of Eq. (8). Since $\gamma_{\varphi\theta_0}$ and $\chi_{\varphi\theta}$ are zero in the axially symmetric case, the shear stress component $\tau_{\varphi\theta}$ is also zero along with $M_{\varphi\theta}$. The derivatives of the sectional moments in Eq. (37), from Eq. (31), become

$$\left.\begin{aligned}
\frac{\partial M_\varphi}{\partial \varphi} &= D_4 \frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} + \epsilon_{\varphi_0}\frac{\partial D_4}{\partial \varphi} + D_5 \frac{\partial \epsilon_{\theta_0}}{\partial \varphi} + \epsilon_{\theta_0}\frac{\partial D_5}{\partial \varphi} - D_6 \frac{\partial \chi_\varphi}{\partial \varphi} \\
&\quad - \chi_\varphi \frac{\partial D_6}{\partial \varphi} - D_7 \frac{\partial \chi_\theta}{\partial \varphi} - \chi_\theta \frac{\partial D_7}{\partial \varphi} - \frac{\partial M_T}{\partial \varphi} \\
\frac{\partial M_\theta}{\partial \varphi} &= D_4 \frac{\partial \epsilon_{\theta_0}}{\partial \varphi} + \epsilon_{\theta_0}\frac{\partial D_4}{\partial \varphi} + D_5 \frac{\partial \epsilon_{\varphi_0}}{\partial \varphi} + \epsilon_{\varphi_0}\frac{\partial D_5}{\partial \varphi} - D_6 \frac{\partial \chi_\theta}{\partial \varphi} \\
&\quad - \chi_\theta \frac{\partial D_6}{\partial \varphi} - D_7 \frac{\partial \chi_\varphi}{\partial \varphi} - \chi_\varphi \frac{\partial D_7}{\partial \varphi} - \frac{\partial M_T}{\partial \varphi}
\end{aligned}\right\} \quad (38)$$

$$
\left.
\begin{aligned}
\frac{\partial^2 M_\varphi}{\partial\varphi^2} = {} & D_4 \frac{\partial^2 \epsilon_{\varphi_0}}{\partial\varphi^2} + 2\frac{\partial \epsilon_{\varphi_0}}{\partial\varphi}\frac{\partial D_4}{\partial\varphi} + \epsilon_{\varphi_0}\frac{\partial^2 D_4}{\partial\varphi^2} + D_5 \frac{\partial^2 \epsilon_{\theta_0}}{\partial\varphi^2} \\
& + 2\frac{\partial \epsilon_{\theta_0}}{\partial\varphi}\frac{\partial D_5}{\partial\varphi} + \epsilon_{\theta_0}\frac{\partial^2 D_5}{\partial\varphi^2} - D_6\frac{\partial^2 \chi_\varphi}{\partial\varphi^2} - 2\frac{\partial D_6}{\partial\varphi}\frac{\partial \chi_\varphi}{\partial\varphi} \\
& - \chi_\varphi \frac{\partial^2 D_6}{\partial\varphi^2} - D_7\frac{\partial^2 \chi_\theta}{\partial\varphi^2} - 2\frac{\partial D_7}{\partial\varphi}\frac{\partial \chi_\theta}{\partial\varphi} - \chi_\theta\frac{\partial^2 D_7}{\partial\varphi^2} \\
& - \frac{\partial^2 M_T}{\partial\varphi^2}
\end{aligned}
\right\} \quad (38)
$$

,

and the derivatives of the neutral surface strains and changes of curvature in Eq. (38), from Eq. (32), become

$$
\left.
\begin{aligned}
\frac{\partial^2 \epsilon_{\varphi_0}}{\partial\varphi^2} = {} & \frac{1}{r}\left(\frac{\partial^3 v}{\partial\varphi^3} + \frac{\partial^2 u}{\partial\varphi^2}\right) \\
\frac{\partial^2 \epsilon_{\theta_0}}{\partial\varphi^2} = {} & \frac{1}{a+r\sin\varphi}\Bigg[\cos\varphi\frac{\partial^2 v}{\partial\varphi^2} + \sin\varphi\frac{\partial^2 u}{\partial\varphi^2} \\
& - \frac{2(a\sin\varphi + r)}{a+r\sin\varphi}\frac{\partial v}{\partial\varphi} + \frac{2a\cos\varphi}{a+r\sin\varphi}\frac{\partial u}{\partial\varphi} \\
& + \frac{\cos\varphi\,(ar\sin\varphi + 2r^2 - a^2)}{(a+r\sin\varphi)^2}\, v - \frac{ar(1+\cos^2\varphi) + a^2\sin\varphi}{(a+r\sin\varphi)^2}\, u\Bigg] \\
\frac{\partial^2 \chi_\varphi}{\partial\varphi^2} = {} & \frac{1}{r^2}\left(\frac{\partial^3 v}{\partial\varphi^3} - \frac{\partial^4 u}{\partial\varphi^4}\right) \\
\frac{\partial^2 \chi_\theta}{\partial\varphi^2} = {} & -\frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial^3 u}{\partial\varphi^3} + \frac{2(r+a\sin\varphi)}{r(a+r\sin\varphi)^2}\frac{\partial^2 u}{\partial\varphi^2} \\
& + \frac{\cos\varphi}{r(a+r\sin\varphi)}\frac{\partial^2 v}{\partial\varphi^2} + \frac{\partial u}{\partial\varphi}\left[\frac{(a^2 - 2r^2\cos^2\varphi)\cos\varphi - r(a+2r\sin\varphi)\sin\varphi\cos\varphi}{r(a+r\sin\varphi)^3}\right] \\
& - \frac{2(r+a\sin\varphi)}{r(a+r\sin\varphi)^2}\frac{\partial v}{\partial\varphi} + v\left[\frac{(a+2r)r\sin\varphi\cos\varphi - (a^2 - 2r^2\cos^2\varphi)\cos\varphi}{r(a+r\sin\varphi)^3}\right]
\end{aligned}
\right\} \quad (39)
$$